

Markov Chain Monte Carlo, with Broken Samplers

Fritz Obermeyer

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<http://fritzo.org/notes/2016/broken.pdf>

But Why?

- ▶ Sub-sampling big data
- ▶ Latency / network failure / worker failure
- ▶ Low-precision numerics / sketching
- ▶ Miscalibrated stochastic hardware

Monte Carlo Sampling

approximates an integral

$$I = \int f(x) dp(x)$$

by a finite average

$$\hat{I} = \frac{1}{T} \sum_{t=1}^T f(X_t) \quad X_t \sim p(-)$$

to minimize squared error

$$\mathbb{E} \left[\left(\hat{I} - I \right)^2 \right] = \underbrace{\mathbb{V} \left[\hat{I} \right]}_{\text{variance}} + \underbrace{\left(\mathbb{E} \left[\hat{I} \right] - I \right)^2}_{\text{bias}}$$

What if your sampler is broken?

Let's say $\theta \sim q(-)$ is a desired parameter distribution

$$p(x) = \int p(x; \theta) d q(\theta)$$

but your sampler is miscalibrated according to $\theta \sim q'(-)$

$$p'(x) = \int p(x; \theta) d q'(\theta)$$

Then reject based on side-observations

If $q'(\theta)/q(\theta)$ is bounded, then we can approximate

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by rejection sampling (error detection).

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If $q'(\theta)/q(\theta)$ is difficult to compute, then we can reject based on summary statistics (Approximate Bayesian Computation).

Markov Chain Monte Carlo

is “Stateful Monte Carlo” with randomized transitions P_t

$$X_0 = \text{fixed}$$

$$X_t \sim P_t(X_{t-1}) \quad \text{for } t \in \{1, \dots, T\}$$

$$f_t = f(X_t)$$

This is a Markov Chain

$$\begin{array}{ccccccc} X_0 & \xrightarrow{P_1} & X_1 & \xrightarrow{P_2} & X_2 & \xrightarrow{P_3} & X_3 & \xrightarrow{P_4} & \dots \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ & & f_1 & & f_2 & & f_3 & & \end{array}$$

Modern MCMC Methods

Optimize the policy $[P_1, \dots, P_T]$ (including duration T)
to minimize error subject to a budget constraint

$$\sum_{t=1}^T \text{cost}(P_t) \leq \text{budget}$$

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Stochastic Gradient Langevin Dynamics
(Welling, Teh 2011)

Approximate Metropolis-Hastings
(Korattikara, Chen, Welling 2014)
(Chen, Fox, Guestrin 2014)

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Idea: Gradually transition from optimization to MCMC
by starting with large noisy steps, then reducing step size.

- ▶ Constant per-step cost $\text{cost}(P_t)$.
- ▶ Policy balances precision (small step size)
with effective speed (large step size).

Approximate Metropolis-Hastings

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- ▶ Variable per-step cost $\text{cost}(P_t)$.
- ▶ Policy balances per-step cost with per-step error.

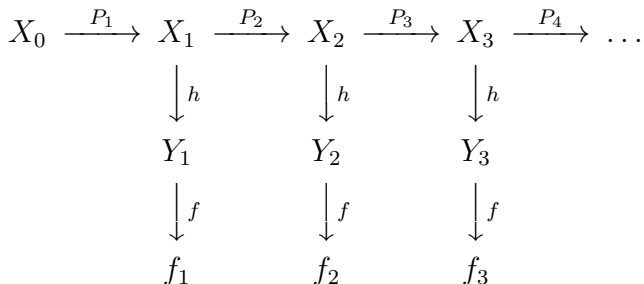
How to optimize an MCMC policy

(in the presence of broken samplers)

Let X_t be hidden sampler state, Y_t be observed. Approximate

$$I \approx \hat{I} = \frac{1}{T} \sum_{t=1}^T f(Y_t)$$

and minimize squared error $\|\hat{I} - I\|^2$ by choosing $[P_1, \dots, P_T]$



What is a Markov Decision Process?

Sequential decision making
with 1-step state history.

Ignores uncertainty in sampler state.

Markov
Chain

Markov
Decision
Process

$$X_0 \xrightarrow{P_1} X_1 \xrightarrow{P_2} X_2 \xrightarrow{P_3} X_3 \xrightarrow{P_4} \dots$$

What is a Partially Observable MDP?

Bayesian approach to sequential decision making.

Model-based approach to Reinforcement Learning.

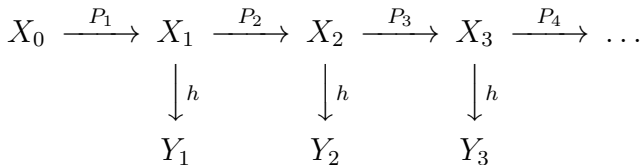
An impossibly intractable 1st step.

Markov Chain

Markov Decision Process

Hidden Markov Model

Partially Observable Markov Decision Process



How to optimize a POMDP?

1-step horizon (Greedy algorithm)

Finite-horizon (Value iteration)

Decaying horizon (Bellman's equation)

1-Step Horizon (Greedy)

Let $f_t = f(Y_t)$. Let $F_t = \sum_{s=1}^t f_s$ be the partial sum.

Suppose you've already sampled $(X_1, Y_1), \dots, (X_{t-1}, Y_{t-1})$.
Now choose P_t to minimize $\mathbb{V}[F_t]$.

Observe that

$$\mathbb{V}[F_t] = \frac{1}{t^2} \mathbb{V}[f_t] + \frac{2(t-1)}{t^2} \text{Cov}[f_t, F_{t-1}] + \text{const.}$$

so that it's much more important to be diverse than precise.

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Problem: This completely ignores future samples.

Value Iteration

1. Solve for final step P_t^* as a function of all previous Y_t , by simply minimizing final variance.
 2. Solve for penultimate iteration P_{t-1}^* by minimizing expected error, assuming optimal solution P_t^* from step 1 will be used after P_{t-1}^* .
 3. etc.
- ... until first step is reached.

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... until first step is reached.

Problem: Doesn't work for variable-cost steps (variable T).

Bellman's Equation for discounted loss

Consider an exponentially decaying objective function, corresponding to the estimator

$$F_t = \frac{F_{t-1} + \gamma(P_t)f(Y_t)}{1 + \gamma(P_t)}, \text{ where } \gamma(P_t) = \frac{\text{cost}(P_t)}{\text{budget}}$$

Assuming a bound k on history relevance, the optimal policy is a fixed point of

$$P^*(Y_{t-k}, \dots, Y_{t-1}) = \arg \min_{P(-)} \mathbb{V}[F_{t-1} + \gamma(P)f(h(P(X_{t-1})))]$$

where the distribution of Y, X, F depends on $P(-)$.

Summary

1. Recognize sequential, stateful nature of problem.
2. Minimize squared error of integral, as in MCMC.
3. Model sampler as a POMDP with hidden noise.
4. Find a policy that balances
Exploitation (estimating the quantity of interest) with
Exploration (calibrating and re-calibrating the sampler).

References

Max Welling, Yee Whye Teh (2011)

“Bayesian Learning via Stochastic Gradient Langevin Dynamics” (pdf)

Anoop Korattikara, Yutian Chen, Max Welling (2014)

“Austerity in MCMC Land: Cutting the Metropolis-Hastings Budget” (pdf)

Tianqi Chen, Emily B. Fox, Carlos Guestrin (2014)

“Stochastic Gradient Hamiltonian Monte Carlo” (pdf)